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Wigner-Type Theorems for Projections

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Abstract The Wigner theorem, in its Uhlhorn's formulation, states that a bijective transformation of the set of all one-dimensional linear subspaces of a complex Hilbert space which preserves orthogonality is induced by either a unitary or an antiunitary operator. There exist in the literature many Wigner-type theorems and the purpose of this paper is to prove in an algebraic setting a very general Wigner-type theorem for projections (idempotent linear mappings). As corollaries, Wigner-type theorems for projections in real locally convex spaces, infinite dimensional complex normed spaces and Hilbert spaces are obtained.

Keywords Orthomodular poset · Lattices of subspaces · Wigner's theorem

1 Introduction

The Wigner theorem states that if H is a complex Hilbert space and S a bijective transformation of the set of all one-dimensional linear subspaces of H which preserves angles between any pair of such subspaces or, in the terminology of quantum mechanics, the transition probability between pure states, then S is induced by a unitary or an antiunitary operator on H. If $\dim H \geq 3$, Uhlhorn [15] improved this result by requiring that S only preserves the orthogonality between the one-dimensional subspaces.

This theorem plays a fundamental role in quantum mechanics and there exist in the literature many generalizations of this result, in particular to indefinite metric spaces [2, 10], von Neumann algebras [9], complex Banach spaces [10], projections of rank one in Banach spaces [10].

The aim of this paper is to prove, in an algebraic setting, a very general Wigner-type theorem in its Uhlhorn's formulation for projections and then to apply it to get Wigner-type theorems for continuous projections in usual topological structures as locally convex spaces,

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normed spaces and Hilbert spaces. In the whole of the paper, by a projection we mean an idempotent linear mapping, sometimes also called a skew projection.

Our method has its origin in [12] where the author describes the form of the automorphisms of the poset of all projections defined on a Hilbert space H by means of automorphisms and antiautomorphisms of the lattice of all closed subspaces of H. This paper shows that it is often more interesting to consider a continuous projection as a pair of closed subspaces, its image and its kernel, rather than a mapping. We specify this approach in Sect. 2 where we recall some results of [3, 4]. In particular, to any lattice L with properties of lattices of closed subspaces is associated a subset $P(L)$ of $L \times L$, called its orthoposet of projections. If L is the lattices of all closed subspaces of a locally convex spaces E then $P(L)$ is isomorphic to the poset of all weakly continuous projections defined on E and a generalization of the main result of [12] to the poset $P(L)$ is given.

The first part of Sect. 3 is devoted to extend a bijection φ of the atoms of a poset of projections $P(L)$ which preserves orthogonality to an automorphism of the orthoposet $P(L)$. Then, by using a result about the automorphisms of lattices of closed subspaces, we prove a Wigner-type theorem for a first class of bijection φ, the ones which transform projections with the same image into projections with the same image. As corollaries, we obtain Wigner-type theorems for projections in real locally convex spaces and infinite dimensional complex normed spaces.

In Sect. 4, by using Hermitian spaces, we obtain a Wigner-type theorem for bijections φ which transform projections with the same image into projections with the same kernel. A corollary gives a Wigner-type theorem for projections in Hilbert spaces.

2 The Orthomodular Poset of Projections of a Lattice

2.1 Definition and Structure

Let a and b be two elements of a lattice L. We say that (a,b) is a modular pair, and write $(a,b)M$, when

$$(x \vee a) \wedge b = x \vee (a \wedge b) \quad \text{for every } x \leq b.$$

If $(a,b)M$ holds in the dual lattice L^* of L, we say that (a,b) is a dual-modular pair and write $(a,b)M^*$.

A lattice L is said to be M-symmetric if $(a,b)M$ implies $(b,a)M$ and M^*-symmetric if L^* is M-symmetric. A symmetric lattice [8] is a lattice which is M-symmetric and M^*-symmetric.

Let L be a symmetric lattice with 0 and 1. The direct product of L and L^* is also a lattice and the set of all elements (a,b) of $L \times L^*$ such that in the lattice L,

$$a \vee b = 1, \qquad a \wedge b = 0, \qquad (a,b)M, \qquad (a,b)M^*,$$

is called the projection poset of L. This poset is denoted by $P(L)$ and any element of $P(L)$ is called a projection of L. If $p = (a,b)$ is a projection then, according to the possible geometrical interpretation of p, a is called the image of p and b the kernel of p.

If (a,b) is a projection of L then, as L is a symmetric lattice, (b,a) is also a projection and we write $(b,a) = (a,b)^\perp$ and define $(a,b) \perp (c,d)$ if $(a,b) \leq (c,d)^\perp$.

Proposition 1 ([4, 11]) *If L is a symmetric lattice with* 0 *and* 1 *then* $(P(L), \leq, \perp)$ *is an orthomodular poset* (*abbreviated OMP*).

The following result shows that our lattice definition of a projection agrees, up to an isomorphism, to the usual concept of a continuous linear projection defined on a locally convex space.

Theorem 1 ([4]) *Let E be a locally convex space and L its lattice of all closed subspaces. The projection orthoposet $P(L)$ is isomorphic to the orthoposet of all weakly continuous linear projections defined on E.*

If E is metrizable then a linear mapping $f : E \to E$ is continuous if an only if f is weakly continuous [13, Chap. IV, 7.4] and so, the orthoposet of projections of the lattice of all closed subspaces of a metrizable locally convex space E is isomorphic to the orthoposet of all continuous linear projections defined on E.

An AC-lattice [8] is an atomistic lattice with the covering property: if p is an atom and $a \wedge p = 0$ then $a \lessdot a \vee p$ that is $a \leq x \leq a \vee p$ implies $a = x$ or $a \vee p = x$.

If L and L^* are AC-lattices, L is called a DAC-lattice. Any DAC-lattice is symmetric and finite-modular [8, Theorem 27.5] and, in general, L_0 will denote the modular lattice of all finite or cofinite elements of the DAC-lattice L. Irreducible complete DAC-lattices of height ≥ 4 are representable by lattices of closed subspaces and many lattices of subspaces are DAC-lattices [8].

An irreducible DAC-lattice L is called a G-lattice [4] if L is complete or if L is modular and complemented. Typical examples of G-lattices are obtained by considering a Hilbert space H: the lattice of all closed subspaces of H is a G-lattice as a complete irreducible DAC-lattice and its sublattice of finite or cofinite dimensional elements is a G-lattice as a complemented modular irreducible DAC-lattice. Any G-lattice of height ≥ 4 has the following properties:

- Any atom has more than one complement;
- If $a \lessdot b$ then there exist different atoms p_1 and p_2 such that $a \vee p_1 = a \vee p_2 = b$;
- Two different atoms have a common complement.

2.2 Automorphisms of an Orthoposet of Projections

The following main result of [3] is a generalization of a theorem of [12] and it gives a description of automorphisms of a projection orthoposet $P(L)$ by means of automorphisms and antiautomorphisms of the lattice L when L is a G-lattice. Moreover, it is proved in [4] that there are exactly two kinds of automorphisms on an orthoposet of projections: the so-called even automorphisms which transform projections with the same image into projections with the same image and the odd automorphisms which transform projections with the same image into projections with the same kernel. This fact generalizes a theorem of [12].

Theorem 2 ([3]) *Let L be a G-lattice of height ≥ 4. For any automorphism ϕ of the orthoposet $P(L)$ there exists*

(1) *an automorphism f of the lattice L such that $\phi((a,b)) = (f(a), f(b))$, $(a,b) \in P(L)$, if ϕ is even,*
(2) *an anti-automorphism g of the lattice L such that $\phi((a,b)) = (g(b), g(a))$, $(a,b) \in P(L)$, if ϕ is odd.*

Conversely, if f is an automorphism of L then $\phi : P(L) \mapsto L \times L^*$, defined by $\phi((a,b)) = (f(a), f(b))$, is an even automorphism of $P(L)$ and if g is an anti-automorphism of L then $\psi : P(L) \mapsto L \times L^*$, defined by $\psi((a,b)) = (g(b), g(a))$, is an odd automorphism of $P(L)$.

Let L be a symmetric lattice. Define on the orthoposet $P(L)$ a binary relation $\perp_1$ by:

$$(a,b) \perp_1 (c,d) \Leftrightarrow b \geq c.$$

It is straightforward to check that:

$$p \perp q \Leftrightarrow p \perp_1 q \text{ and } q \perp_1 p.$$

The binary relation $\perp_1$ allows one to characterize even automorphisms among automorphisms of an orthoposet of projections.

Proposition 2 *Let $P(L)$ be the orthoposet of projections of a G-lattice L of height ≥ 4 and $\Phi : P(L) \to P(L)$.*

(1) *If Φ preserves $\perp_1$ in both directions then Φ also preserves $\perp$.*
(2) *The mapping Φ is an even automorphism of $P(L)$ if and only if Φ is an automorphism which preserves $\perp_1$ in both directions.*

Proof (1) Let p and q be two projections. We can write:

$$\begin{aligned} p \perp q &\Leftrightarrow p \perp_1 q \text{ and } q \perp_1 p \Leftrightarrow \Phi(p) \perp_1 \Phi(q) \text{ and } \Phi(q) \perp_1 \Phi(p) \\ &\Leftrightarrow \Phi(p) \perp \Phi(q). \end{aligned}$$

(2) Assume that Φ is an even automorphism and let $f : L \to L$ be the automorphism of L such that $\Phi(a,b) = (f(a), f(b))$. For two projections (a,b) and (c,d), we have:

$$\begin{aligned} (a,b) \perp_1 (c,d) &\Leftrightarrow b \geq c \Leftrightarrow f(b) \geq (f(c) \Leftrightarrow (f(a), f(b)) \perp_1 (f(c), f(d)) \\ &\Leftrightarrow \Phi(a,b) \perp_1 \Phi(c,d). \end{aligned}$$

Now assume that Φ is an odd automorphism and let g be the antiautomorphism of L satisfying $\Phi(a,b) = (g(b), g(a))$. Let α and β be two different atoms of L and let γ be a common complement for α and β. The pairs (α,γ) and (γ,β) are projections and $(\alpha,\gamma) \perp_1 (\gamma,\beta)$. Since α and β are incomparable, $g(\alpha)$ and $g(\beta)$ are also incomparable and $\phi(\alpha,\gamma) = (g(\gamma), g(\alpha)) \perp_1 (g(\beta), g(\gamma)) = \phi(\gamma,\beta)$ does not hold true. The odd automorphism ϕ does not preserve $\perp_1$. □

Remark

(1) Consider the binary relation $\perp_2$ define on $P(L)$ by

$$(a,b) \perp_2 (c,d) \Leftrightarrow a \leq d.$$

We have

$$p \perp q \Leftrightarrow p \perp_1 q \text{ and } p \perp_2 q$$

and Φ is an odd automorphism of $P(L)$ if and only if Φ is an automorphism such that

$$p \perp_2 q \Leftrightarrow \Phi(q) \perp_2 \Phi(p).$$

(2) If p and q are linear projections defined on a vector space then $p \perp_1 q$ means $\mathrm{Im}\, q \subset \ker p$ and is equivalent to $pq = 0$. Therefore, the automorphism Φ is even if and only if $pq = 0 \Leftrightarrow \Phi(p)\Phi(q) = 0$ and, if Φ is odd, $pq = 0 \Leftrightarrow \Phi(q)\Phi(p) = 0$.

3 A Wigner-Type Theorem for Projections in DAC-Lattices

In this section, we will prove a Wigner-type theorem for projections of a DAC-lattice L. Since $P(L)$ is only an orthoposet, some preliminary results are necessary to extend a bijection φ of the atoms of $P(L)$ which preserves orthogonality to an automorphism of $P(L)$.

3.1 Preliminary Results

A projection $p = (a, b)$ is called finite if $a = 0$ or is a join of a finite number of atoms and the height of a [8, Definition 8.5], is called the height of p. If b is a meet of a finite number of coatoms then p is called a cofinite projection. For linear projections defined on a vector space, a finite projection is also called a rank-finite projection and its height is the dimension of its image.

Proposition 3 *Let L be a G-lattice and let L_0 be the modular G-lattice of all finite or cofinite elements of L. In $P(L_0)$, any finite projection, different from* 0, *is a join of a finite orthogonal family of atoms.*

Proof The proof is by induction on the height of a finite projection of $P(L_0)$. Assume that any finite projection of height $n \geq 1$ is a join of a finite orthogonal family of n atoms and let P be a projection of height $n + 1$. We can write $P = (a, b) = (a' \vee p, b)$ with a' an element of L_0 of height n and p an atom such that $p \wedge a' = 0$. Let $q = a' \vee b$. We have $p \vee q = a \vee b = 1$. If $p \leq q$ then $q = 1$ and thus a and a' are perspective with b as a common complement. This contradicts the modularity of L_0 since the heights of a and a' are different. Therefore $p \wedge q = 0$, $q \lessdot p \vee q = 1$ and q is a coatom.

Let $b' = b \vee p$. We have $b \lessdot b'$ and $a' \vee b' = a' \vee b \vee p = a \vee b = 1$. Since $b' \nleq a$, $a \wedge b' < b'$ and $a \wedge b' \leq b$. We have $a \wedge b' = a \wedge a' \wedge b' \leq a \wedge b = 0$ and thus $a' \wedge b' = 0$ and (a', b') is a finite projection of height n. As $(a', b') \perp (p, q)$ we can write:

$$(a', b') \vee (p, q) = (a' \vee p, b' \wedge q) = (a, b' \wedge q) = (a, b)$$

since, by using the modularity of L_0 and $b \leq q$,

$$b' \wedge q = (b \vee p) \wedge q = b \vee (p \wedge q) = b \vee 0 = b.$$

Thus, any projection of height $n + 1$ is a join of a finite orthogonal family of $n + 1$ atoms and the proposition is proved by induction. □

By using the dual lattice L^*, one can obtain a similar result about cofinite projection of $P(L)$. A family (p_i) of orthogonal atoms of $P(L_0)$ such that $p = \bigvee p_i$ is called a basis for p.

3.2 Extension of the Bijection

Theorem 3 *Let L be a G-lattice of height ≥ 4. If φ is a bijection of the set of all atoms of $P(L)$ satisfying for any atoms p, q,*

$$p \perp q \Leftrightarrow \varphi(p) \perp \varphi(q)$$

then φ can be extended to an automorphism ϕ of the orthomodular poset $P(L)$.

Proof In a first step of the proof, we will extend φ to an automorphism of the orthomodular poset of all finite or cofinite projections.

Let $\bigvee_{i=1}^m p_i$ and $\bigvee_{i=1}^n q_i$ be two joins of orthogonal atoms of $P(L_0)$. Assume that $\bigvee_{i=1}^n q_i \leq \bigvee_{i=1}^m p_i$. For any atom $p = \varphi(q)$ of $P(L)$ we can write:

$$\begin{aligned}
p = \varphi(q) \perp \bigvee_{i=1}^m \varphi(p_i) &\Leftrightarrow \text{for all } i \in [1,m],\ \varphi(q) \perp \varphi(p_i) \\
&\Leftrightarrow \text{for all } i \in [1,m],\ q \perp p_i \\
&\Leftrightarrow q \perp \bigvee_{i=1}^m p_i \\
&\Rightarrow q \perp \bigvee_{i=1}^n q_i \\
&\Leftrightarrow \text{for all } i \in [1,n],\ q \perp q_i \\
&\Leftrightarrow \text{for all } i \in [1,n],\ p = \varphi(q) \perp \varphi(q_i) \\
&\Leftrightarrow p \perp \bigvee_{i=1}^n \varphi(q_i).
\end{aligned}$$

Since $P(L)$ is atomistic [4, Lemma 6], we have $\bigvee_{i=1}^n \varphi(q_i) \leq \bigvee_{i=1}^m \varphi(p_i)$. Since φ^{-1} shares the properties of φ, $\bigvee_{i=1}^n \varphi(q_i) \leq \bigvee_{i=1}^m \varphi(p_i)$ implies $\bigvee_{i=1}^n q_i \leq \bigvee_{i=1}^m p_i$. In particular, $\bigvee_{i=1}^n q_i = \bigvee_{i=1}^m p_i$ is equivalent to $\bigvee_{i=1}^n \varphi(q_i) = \bigvee_{i=1}^m \varphi(p_i)$.

If $(p_i)_{i\in[1,m]}$ is a basis for the finite projection $p \neq 0$, we can define ϕ by $\phi(p) = \bigvee_{i=1}^m \varphi(p_i)$ and complete this definition by $\phi(0) = 0$. Now if p is a cofinite projection let $\phi(p) = (\phi(p^\perp))^\perp$.

Note that, for any finite or cofinite projection p, $\phi(p)^\perp = \phi(p^\perp)$.

In order to prove that the mapping ϕ is onto, let $p \in P(L_0)$. If $p = 0$, $p = \phi(0)$ and if p is finite and different from 0, then $p = \bigvee_{1=1}^n p_i$, with p_i atom of $P(L_0)$, and we have $p = \phi(\bigvee_{1=1}^n \varphi^{-1}(p_i))$. If p is cofinite then there exits q such that $p^\perp = \phi(q)$ and $p = (\phi(q))^\perp = \phi(q^\perp)$.

Now, we will prove that ϕ is an automorphism of the poset $P(L_0)$. Let p and q be two elements of $P(L_0)$. If (p,q) is a pair of finite projections or a pair of cofinite projections, it is clear that:

$$p \leq q \Leftrightarrow \phi(p) \leq \phi(q).$$

Now if p is an atom of $P(L_0)$ and q is cofinite, $q = \bigwedge_{i=1}^n q_i^\perp$, q_i atom of $P(L_0)$, we have:

$$
\begin{aligned}
p \le q &\Leftrightarrow \forall i \in [1,n],\ p \le q_i^{\perp} \Leftrightarrow \forall i \in [1,n],\ p \perp q_i \\
&\Leftrightarrow \forall i \in [1,n],\ \varphi(p) \perp \varphi(q_i) \Leftrightarrow \forall i \in [1,n],\ \varphi(p) \le \varphi(q_i)^{\perp} \\
&\Leftrightarrow \varphi(p) \le \bigwedge_{i=1}^{n} \varphi(q_i)^{\perp} = \left(\bigvee_{i=1}^{n} \varphi(q_i)\right)^{\perp} = \left(\phi\left(\bigvee_{i=1}^{n} q_i\right)\right)^{\perp} = \phi(q^{\perp})^{\perp} = \phi(q).
\end{aligned}
$$

By using the definition of ϕ this equivalence is also true if p is a finite projection and thus ϕ is an automorphism of the poset $P(L_0)$. As $\phi(p)^{\perp} = \phi(p^{\perp})$, ϕ is an automorphism of the orthomodular poset $P(L_0)$.

Now we will extend ϕ to $P(L)$. As L_0 is a G-lattice of height ≥ 4 then, by Theorem 2, there exists an automorphism f of the lattice L_0 such that $\phi((a,b)) = (f(a), f(b))$, $(a,b) \in P(L_0)$, if ϕ is even or there exist an antiautomorphism g of L_0 such that $\phi((a,b)) = (g(b), g(a))$ if ϕ is odd. By Lemma 1 of [3], f can be extended to an automorphism of L and we define ϕ for $(a,b) \in P(L)$ by $\phi((a,b)) = (f(a), f(b))$ if $\phi : P(L_0) \to P(L_0)$ is even. The extension of ϕ if $\phi : P(L_0) \to P(L_0)$ is odd is similar.

Finally, by Proposition 6 of [4], ϕ is an automorphism of the orthomodular poset $P(L)$ and the proof is complete. □

Corollary 1 *Let L be a G-lattice. For any bijection f of $P(L)$ into itself the following statements are equivalent*:

(1) *f preserves the orthogonality relation in both directions*;
(2) *the restriction of f to the set of all atoms of $P(L)$ is a bijection of this set which preserves the orthogonality relation in both directions*
(3) *f is an automorphism of the orthomodular poset $P(L)$.*

Proof The proof is easy by using the previous proposition, Proposition 6 of [4] saying that any order automorphism f of $P(L)$ satisfies $f(p^{\perp}) = f(p)^{\perp}$ and the fact that, in any orthocomplemented lattice T,

$$a \le b \Leftrightarrow \forall c \in T,\ c \perp b \Rightarrow c \perp a. \qquad \square$$

In [14], the author consider the set $P(X)$ of all projections defined on a Banach space X and he proves that $\Phi : P(X) \to P(X)$ is an order automorphism of $P(X)$ if and only if Φ preserves the orthogonality relation in both directions.

Example Let E be an infinite dimensional vector space and let L be the G-lattice of all the subspaces of E. Since E is infinite dimensional, L has no antiautomorphism [1, p. 111], and therefore $P(L)$ possesses no odd automorphism. If φ is a bijection of the set of all atoms of $P(L)$ (i.e. a bijection of the set of all projections of rank one) which preserves orthogonality then, by the First Fundamental Theorem of projective geometry [1] and Theorems 2 and 3, there exists a semi-linear bijection $s : E \to E$ such that for any projection $p = (a,b)$, $\Phi(p) = (s(a), s(b))$, where Φ is the extension of φ to $P(L)$. If p is identified to a linear projection then $\Phi(p) = sps^{-1}$ since $\mathrm{Im}\, sps^{-1} = s(\mathrm{Im}\, p)$ and $\mathrm{Ker}\, sps^{-1} = s(\mathrm{Ker}\, p)$.

3.3 Automorphisms of Lattices of Closed Subspaces

Now, in order to obtain, by means of Theorems 2 and 3, a Wigner-type theorem for continuous projections defined on certain topological vector spaces, its is necessary to know automorphisms and antiautomorphisms of lattices of closed subspaces.

Let K be a field, E a left vector space over K, F a right vector space over K. If there exists a non degenerate bilinear form $\mathcal{B}$ on $E \times F$, we say that (E, F) is a pair of dual spaces [5]. Since the form is nondegenerate, F can be identified to a subspace of the algebraic dual E^* of E and E to a subspace of F^*.

For a subspace A of E, we put

$$A^{\perp} = \{y \in F \mid \mathcal{B}(x, y) = 0 \text{ for every } x \in A\}.$$

Similarly, let

$$B^{\perp} = \{x \in E \mid \mathcal{B}(x, y) = 0 \text{ for every } y \in B\}$$

for every subspace B of F. A subspace A of E is called F-closed if $A = A^{\perp\perp}$ and the set of all F-closed subspaces, denoted by $L_F(E)$ and ordered by set-inclusion, is a complete irreducible DAC-lattice [8, Theorem 33.4] and therefore a G-lattice. Conversely, for any irreducible complete DAC-lattice L of height ≥ 4, there exists a pair (E, F) of dual spaces such that L is isomorphic to the lattice of all F-closed subspaces of E [8, Theorem 33.7], [7, §10.3].

The set of all E-closed subspaces of F is similarly defined and is also a complete DAC-lattice. The two DAC-lattices $L_F(E)$ and $L_E(F)$ are dual isomorphic by the mapping $A \to A^{\perp}$ [8, Theorem 33.4].

Let (E, F) be a pair of dual spaces. The linear weak topology on E, denoted by $\sigma(E, F)$, is the linear topology defined by taking $\{G^{\perp} \mid G \subset F, \ \dim G < \infty\}$ as a basis of neighborhoods of 0. If F is interpreted as a subspace of the algebraic dual of E then a subbasis of neighborhoods of 0 consists of kernels of elements of F.

The linear weak topology on F, noted $\sigma(F, E)$, is defined in the same way. The space F can be interpreted as the topological dual of E for the $\sigma(E, F)$ topology and E as the topological dual of F for the $\sigma(F, E)$ topology. Equipped with their linear weak topologies, E and F are topological vector spaces [7, §10.3] if the topology on K is discrete.

In the following proposition, we generalize the First Fundamental Theorem of projective geometry related to automorphisms of the lattice of all subspaces of a vector space to lattices of closed subspaces.

Proposition 4 ([3]) *Let (E_1, F_1) and (E_2, F_2) be two pairs of dual spaces over the fields K_1 and K_2. If there exists an isomorphism ψ of the lattice $L_{F_1}(E_1)$ onto the lattice $L_{F_2}(E_2)$ then K_1 and K_2 are isomorphic fields and there exists a semi-linear bijection $s : E_1 \mapsto E_2$ such that, for every F_1-closed subspace M of E_1, $\psi(M) = s(M)$.*

In the case of lattices of all subspaces of a vector space, any semi-linear bijection induces a lattice automorphism. For lattices of closed subspaces, only continuous semi-linear bijections are allowed and more precisely we have:

Proposition 5 ([3]) *Let (E_1, F_1) and (E_2, F_2) be two pairs of dual spaces over the same field. If E_1 and E_2 are equipped, respectively, with the $\sigma(E_1, F_1)$-topology and the $\sigma(E_2, F_2)$-topology then, for every semi-linear bijection $s : E_1 \mapsto E_2$, the following statements are equivalent.*

(1) *The bijection s is bicontinuous (i.e. both s and s^{-1} are continuous).*
(2) *$H \in L_{F_1}(E_1) \mapsto s(H)$ is a bijection from the set of all F_1-closed hyperplanes of E_1 onto the set of all F_2-closed hyperplanes of E_2.*
(3) *$M \in L_{F_1}(E_1) \mapsto s(M)$ is an isomorphism from the lattice $L_{F_1}(E_1)$ onto $L_{F_2}(E_2)$.*

3.4 A Wigner-Type Theorem in Topological Vector Spaces in the Even Case

A bijection φ of the atoms of a projection lattice $P(L)$ is called an even bijection if φ transform projections with the same image into projections with the same image. If an even bijection φ extends to an automorphism of $P(L)$ then this automorphism is even.

Theorem 4 *Let φ be an even bijection of the atoms of the projection orthoposet of the G-lattice of all F-closed subspaces of a dual pair (E, F). If φ preserves orthogonality,*

$$p \perp q \Leftrightarrow \varphi(p) \perp \varphi(q),$$

then φ extends to an even automorphism Φ of the orthoposet $P(L_F(E))$ and there exists a weakly bicontinuous semi-linear bijection $s : E \to E$ such that, for any projection $p = (a, b)$

$$\Phi(p) = (s(a), s(b)).$$

Proof By using Theorem 3, φ extends to an automorphism of the orthoposet of projections of the G-lattice $L_F(E)$. Since Φ is even there exists by Theorem 2 an automorphism f of $L_F(E)$ such that $\Phi(a, b) = (f(a), f(b))$. By using Propositions 4 and 5, f is induced by a weakly bicontinuous semi-linear bijection s. □

By particularizing the dual pair (E, F), we can obtain several corollaries of this theorem. In these corollaries, a projection is now considered as a linear mapping and thus the conclusion of the previous theorem becomes $\Phi(p) = sps^{-1}$ since $\operatorname{Im} sps^{-1} = s(\operatorname{Im} p)$ and $\operatorname{Ker} sps^{-1} = s(\operatorname{Ker} p)$.

First, we recall that a linear mapping f, defined on a locally convex space E over $K = \mathbb{R}$ or $\mathbb{C}$, is weakly continuous if and only f is continuous with respect to the linear weak topology $\sigma(E, E')$ [7, 20.4] where E' denotes the topological dual of E and, if $K = \mathbb{R}$ then a semi-linear mapping is linear since the identity is the only automorphism of the field $\mathbb{R}$.

Corollary 2 *Let E be a real locally convex space. If φ is a even bijection of the set of all rank-one weakly continuous projections which preserves orthogonality in both directions then*:

(1) *φ extends to an even automorphism of the orthoposets* Proj(E) *of all weakly continuous projections of E*,
(2) *there exists a weakly bicontinuous linear bijection $s : E \to E$ such that, for any projection p, $\Phi(p) = sps^{-1}$*.

If E is metrizable then Proj(E) *is the orthoposet of all continuous projections and s is bicontinuous*.

For the last claim of this corollary, we have used the fact that weakly continuous linear mappings between metrizable spaces are continuous [13, Chap. IV, 3.4 and 7.4].

If $K = \mathbb{C}$ then the automorphism τ associated to the semi-linear bijection s of Theorem 4 can be not continuous (In a locally convex space over a field K, the topology on K is not the discrete one but is defined by means of the modulus) and an hypothesis stronger than locally convex seems necessary.

Corollary 3 *Let E be an infinite-dimensional complex normed spaces. If φ is a even bijection of the set of all rank-one continuous projections which preserves orthogonality in both directions then φ extends to an even automorphism Φ of the orthoposet of all continuous projections and there exists a bicontinuous linear or conjugate linear bijection $s : E \to E$ such that, for any projection p, $\Phi(p) = sps^{-1}$.*

Proof Let s be the semi-linear bijection obtained by using Theorem 4. Since s is continuous for the weak linear topology, s carries orthogonally closed hyperplanes to orthogonally closed hyperplanes (Proposition 5). But orthogonally closed subspaces of E agree with topologically closed subspaces [7, §20, 3(2)] and, by using a result of [6], s is either linear or conjugate linear. A linear mapping on a metrizable space E is continuous if and only if this mapping is continuous for the weak linear topology $\sigma(E, E')$ and the generalization of this result to a conjugate linear mapping is easy. Thus, s is continuous and by, using a similar proof, s^{-1} is also continuous. □

Remark In [10], the same result is proved for real or complex Banach spaces.

4 A Wigner-Type Theorem for Odd Automorphisms

There exist odd automorphisms on an orthoposet of projections $P(L)$ if and only if the lattice L has antiautomorphisms. If L is the lattice of all subspaces of a vector space E then the situation is clear: if E is infinite dimensional, L has no antiautomorphism and otherwise, the anti-automorphisms of L are defined by means of non-degenerated semi-bilinear forms [8, 16].

Now, if L is the lattice of all closed subspaces of a topological space, we don't know the form of the antiautomorphisms of L and prefer to study the case of Hermitian spaces. Some definitions and results are necessary.

Let K be a field with an involutorial antiautomorphism $\sigma : \lambda \to \bar{\lambda}$. A left vector space E over K is called a Hermitian space if there exists a σ-semi-bilinear form $B : E \times E \to K$ satisfying the two conditions:

- $B(x, y) = \overline{B(y, x)}$,
- $B(x, x) = 0$ implies $x = 0$.

The pair (E, E) is a dual pair (the second E is equipped with the right scalar product $x * \lambda = \bar{\lambda}x$) and all the E-closed subspaces of E form an irreducible complete orthocomplemented AC-lattice denoted by $\mathcal{C}(E)$. Conversely, any irreducible complete orthocomplemented AC-lattice of height ≥ 4 is isomorphic to the orthocomplemented lattice of all closed subspace of a Hermitian space [8, Theorem 34.5].

Remark that a complete orthocomplemented AC-lattice is a G-lattice.

If L is an orthocomplemented AC-lattice then $a \to a^\perp$ is an isomorphism from L onto L^* and therefore, for any $(a, b) \in L$,

$$(a, b)M \Leftrightarrow (a^\perp, b^\perp)M^* \quad \text{and} \quad (a, b)M^* \Leftrightarrow (a^\perp, b^\perp)M.$$

Therefore if $p = (a, b) \in P(L)$ then $(b^\perp, a^\perp)$ is also a projection called the adjoint of p and denoted p^*.

Proposition 6 *Let L be an orthocomplemented AC-lattice. The mapping $Ad : p \to p^*$ is an odd involutary automorphism of the orthoposet $P(L)$.*

Proof The mapping Ad is bijective and

$$\begin{aligned}(a,b)\le(c,d) &\Leftrightarrow a\le c \text{ and } b\ge d\\ &\Leftrightarrow a^{\perp}\ge c^{\perp} \text{ and } b^{\perp}\le d^{\perp}\\ &\Leftrightarrow (b^{\perp},a^{\perp})\le(d^{\perp},c^{\perp})\\ &\Leftrightarrow (a,b)^{*}\le(c,d)^{*}.\end{aligned}$$

Moreover

$$\begin{aligned}(a,b)\perp_2(c,d) &\Leftrightarrow a\le d\\ &\Leftrightarrow d^{\perp}\le a^{\perp}\\ &\Leftrightarrow (d^{\perp},c^{\perp})\perp_2(b^{\perp},a^{\perp})\\ &\Leftrightarrow (c,d)^{*}\perp_2(a,b)^{*}.\end{aligned}$$

Thus, Ad is an odd automorphism of $P(L)$ and it is clear that Ad is involutary. □

A Wigner-type theorem for projections in Hermitian space has the following form in the case of odd bijections.

Theorem 5 *Let E be an Hermitian space. If φ is an odd bijection of the atoms of the projection orthoposet of $\mathcal{C}(E)$ which preserves orthogonality,*

$$p\perp q\Leftrightarrow\varphi(p)\perp\varphi(q),$$

then φ extends to an odd automorphism Φ of the orthoposet $P(\mathcal{C}(E))$ and there exists a weakly bicontinuous semi-linear bijection $s:E\to E$ such that, for any projection $p=(a,b)$

$$\Phi(a,b)=(s(b^{\perp}),s(a^{\perp}))$$

or, in the language of linear projections,

$$\Phi(p)=sp^{*}s^{-1}.$$

Proof By using Theorem 3, φ extends to an automorphism Φ of the orthoposet of projections of the DAC-lattice $\mathcal{C}(E)$. Since Φ is odd, $p\in P(\mathcal{C}(E))\to\Phi(p^{*})$ is even and there exists by Theorem 2 an automorphism f of $\mathcal{C}(E)$ such that $\Phi((a,b)^{*})=(f(a),f(b))$ for any $p=(a,b)\in P(\mathcal{C}(E))$. That means $\Phi(b^{\perp},a^{\perp})=(f(a),f(b))$ or also $\Phi(a,b)=(f(b^{\perp}),f(a^{\perp}))$. By using Proposition 5, f is induced by a weakly bicontinuous semi-linear bijection s and we have $\Phi(a,b)=(s(b^{\perp}),s(a^{\perp}))$. In the language of linear projections, $\Phi(p)=sp^{*}s^{-1}$ since $\operatorname{Im}p^{*}=(\operatorname{Ker}p)^{\perp}$. □

Remark If E is a real Hilbert space then s is a linear and bicontinuous bijection and if E is an infinite dimensional complex Hilbert space then s is a linear or conjugate linear bicontinuous bijection. In [12], the main result states that if Φ is an automorphism of the orthomodular poset of projections in a complex Hilbert space H then there exists a semi-linear linear bijection $s:H\to H$ such that $\Phi(p)=sps^{-1}$ or $\Phi(p)=sp^{*}s^{-1}$. Moreover, if H is infinite dimensional, then s is a linear or conjugate linear bicontinuous bijection.

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